EC3324 Autumn Lecture #07
Strategic Games with Imperfect Information

- Reading
  - Osborne, Chapters 9, 3.5
- Learning outcomes
  - understand the concept of a Bayesian game
  - find the equilibrium of a Bayesian game
  - be familiar with simple auction formats

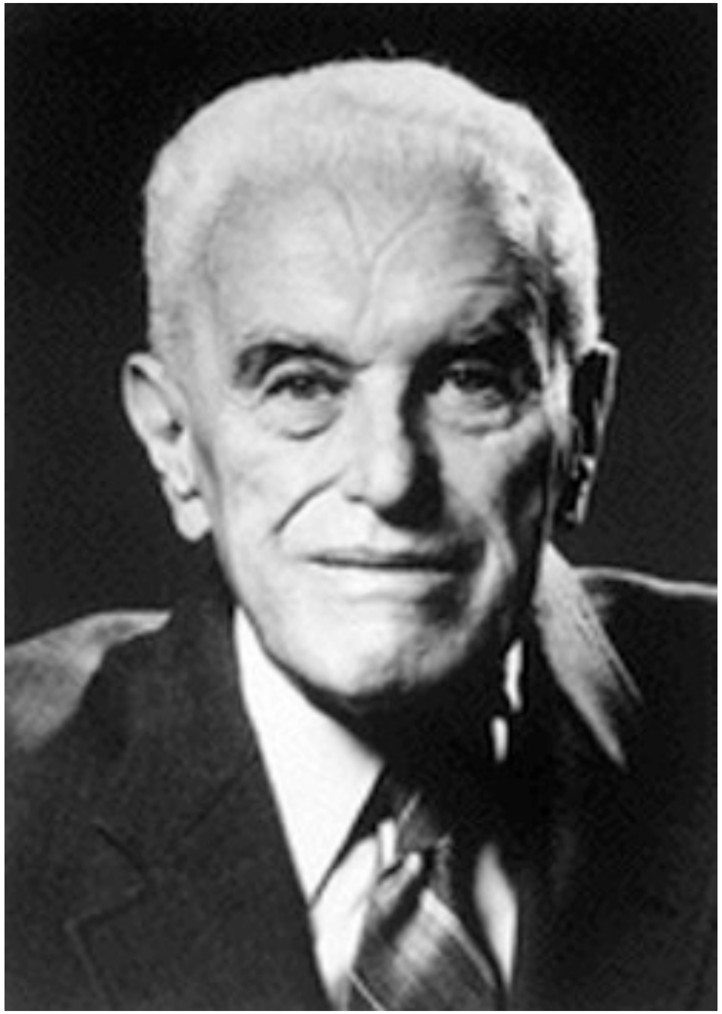
Incomplete Information and Bayesian Games

- So far, we have considered games of **complete information:** all players know the preferences of all others

- We now consider situations, where players have **incomplete information:** they do not know some relevant characteristic of other players. This may include the payoffs, the actions, and the beliefs

- Following Harsanyi's 1967-68 trilogy (led to 1994 Nobel Prize) we model such situations as games of **imperfect information** => **Bayesian games**
  - See R. Myerson's (another Nobel prize here) great essay for details (http://home.uchicago.edu/~rmyerson/research/harsinfo.pdf)
- In a Bayesian game there are different **types** of the players (more precisely of at least one of the players) and players know their own type but not the type of the other players

Nobel Prize in Economics, 1994



- Reinhard Selten (left) - subgame perfection, John Harsanyi (right) - imperfect information and…

John F. Nash (not only developed the homonymous equilibrium concept but has also beaten the emperor of Rome in “Gladiator”)

Example: Prisoner's dilemma

If Player 1 faces a selfish Player 2, then the game is

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	2,2	0,3
	D(efect)	3,0	1,1

But Player 2 might also be a **cooperative type**, who likes to cooperate, then the payoff matrix could be, for example

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	2,3	0,2
	D(efect)	3,1	1,0

In either case, however, D is a dominant strategy for Player 1

Example: Prisoner's dilemma

Now assume that Player 1 herself is not selfish, but a **conditional cooperator**, i.e. she likes to cooperate as long as others do.

Then if Player 2 is selfish, the matrix is, e.g.,

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	3,2	0,3
	D(efect)	2,0	1,1

But if Player 2 is cooperative, the payoff matrix is, e.g.

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	3,3	0,2
	D(efect)	2,1	1,0

It now matters which type of Player 2 Player 1 believes to face

Example: Prisoner's dilemma

- Intuitively, in a Nash-equilibrium of such a game, each **type** of Player 2 should play a best response against Player 1 and Player 1 a best response against his belief about the type he faces and the actions of both types
- Let the probability for Player 2 to be type 1 (selfish) be p
- Then the expected payoff for Player 1 is

		Player 2 $(a_2(t_1), a_2(t_2))$			
		(C,C)	(C,D)	(D,C)	(D,D)
Player 1	C	3	$3p$	$3(1-p)$	0
	D	2	$2p+(1-p)$	$p+2(1-p)$	1

- In Tables on previous slides, we see that for type 1 of Player 2, D is dominant, and for type 2, C is dominant.
- Player 1's best response to (D,C) is C if $p < 1/2$ and D if $p > 1/2$
- Nash equilibria: (C,(D,C)) if $p < 1/2$ and (D,(D,C)) if $p > 1/2$
- For $p = 1/2$, Player 1 is indifferent, so we get arbitrary mixing

Bayesian games

A **Bayesian game** captures the ideas from the above example. It consists of

- a set of **players**
- a set of **states**

and for each player

- a set of **actions**
- a set of **signals**, where each state is assigned a signal (though several states may be assigned the same signal)
- for each signal a **belief** about the states consistent with this signal
- a **Bernoulli payoff function** over pairs (a,w) where a is an action profile and w a state (so that preferences over lotteries over such pairs are represented by the expected payoff)

We can understand this as a player "nature" drawing the types and the players not being perfectly informed about nature's moves

This links a Bayesian game to the analysis of extensive games with imperfect information to be discussed in lecture 8.

Application to the Prisoner's Dilemma

- Players and actions as standard Prisoner's Dilemma
- Set of two **states**, characterized by Player 2's preferences: {*selfish*, *cooperative*}
- Player 1 receives the same signal in both states, so the signal is not informative
- Player 2 receives a signal informing her about her preferences
- Player 1 assigns probability p to state *selfish* , $1 - p$ to *cooperative*
- The payoffs for Player 2 depend on the state (first table for *selfish*, second table for *cooperative*)
- A player has to choose an action for each possible signal
- We can identify a **type** of a player with each possible signal that the player could receive
  - In equilibrium each type has to choose optimally
- Here Player 1 has one type and Player 2 has two types
  - We can treat this as a 3-Player game

Nash equilibrium of a Bayesian game

A **Nash equilibrium of a Bayesian game** (Bayes-Nash equilibrium) is a Nash equilibrium of the strategic game with

- **Players:** all types of all players in the **Bayesian game,** i.e. the set of all pairs (i,t_i) where i is a player in the Bayesian game and t_i a possible signal (type) of i
- **Actions:** For each player (i,t_i) the set of actions is the set of actions of player i in the Bayesian game
- **Preferences:** given by the expected payoff over all states given the beliefs (conditional on the signal t_i) and the actions of all players (where the state determines their signal, which in turn determines their action given their strategy)

Note: while a player knows her own type, we nevertheless need to consider the actions of *all types*, because they impact on the other players

- For each type of a player, we can treat the other types' action as given, because they have no influence on the given type's payoff, so technically we can treat them like other players

Finding a Bayes-Nash equilibrium:

- Calculating Bayes-Nash equilibria can be messy, in particular in games with $n > 2$ players who all have several types
- But often, **dominance** helps us:
  - First, we need to reformulate the **dominance criterion**: strategy *r* dominates *s* if *r* yields a higher expected payoff against any **type-dependent strategy vector** of the other(s)
  - Now if a **type** of a player has a **dominant strategy**, he has to play it (see the prisoner's dilemma for the selfish or cooperative type)
  - Everybody knows that, so we can fix this strategy for this type and continue
  - Similarly, if a type of a player has a **dominated** strategy, we can eliminate this as usual and continue (this is particularly useful if this strategy is dominated for **all types** of a particular player, because then the other players know this strategy will not be played no matter which type they face)

Bayesian games can also be used to capture other types of uncertainty, e.g. about the other players' knowledge.

	State a	
	L	R
L	2,2	0,0
R	3,0	1,1

	State b	
	L	R
L	2,2	0,0
R	0,0	1,1

	State c	
	L	R
L	2,2	0,0
R	0,0	1,1

Two signals for P1: (**a**) and (**b** or **c**). If he receives (**b** or **c**), then beliefs are Pr(**b**|(**b** or **c**)) = ¾, Pr(**c**|(**b** or **c**)) = ¼

Two signals for P2: (a or b) and (c). If he receives (a or b), then beliefs are Pr(**a**|(**a** or **b**)) = ¾, Pr(**b**|(**a** or **b**)) = ¼

Now let the state be **c**. Then both players know both preferences, but P1 does not know that P2 knows P1's preferences

While there are two equilibria in the game corresponding to state **c**, in the equilibrium of this Bayesian game, all play R:

In state **a**, R is dominant for P1. So when P1 gets signal (**a**), he will play R. When P2 gets signal (**a** or **b**), then given beliefs, R yields higher expected payoff no matter what P1 does in state **b**, so P2 chooses R in both states **a** and **b**

If P1 gets signal (**b** or **c**), given beliefs and P2's action R in **b**, R is best response

Finally, if P2 gets signal (**c**), given P1's action in **c**, R is best response

Imperfect Information in Economics Experiments

Participants in economics experiments often effectively play a Bayesian game:

- Experimental results often **differ** from the theoretical prediction
  - This can (sometimes) be captured by players having ***other regarding preferences***
- Frequently **heterogeneity** in **actions** and thus apparently in **preferences** (or in beliefs) is observed
- Thus in experiments, the participants are typically in a situation where they do **not** know the **type** of the other player(s), but could know the **distribution**

  (in particular if they play repeatedly with random matching)
- This is captured by a Bayesian game

Example: guessing game (Nagel, 1995)

- N players
- Submit a number [0,100]
- Winner is the one closest to 2/3 times the average of all numbers
- Nash equilibrium?
- Do people play it?
- Explanation: Bayesian game where opponents' type can be {*rational*, *irrational*}
- Many games with similar property: centipede, travelling salesman, auctions with resale

Auctions

- An Auction is a mechanism to sell (or buy) an object where potential buyers make bids and who obtains the object and the price are solely determined by these bids.

Implications:

- Auctions are **anonymous:** only bids matter, not who made them (this is violated by the tie-breaking rule in Osborne 3.5)
- Auctions are **universal:** any arbitrary object can be sold in an auction
- Auctions are useful when the seller is not sure about the valuations of the buyers for the object. Otherwise he can just offer it to the bidder with the highest willingness to pay
  - Auctions are about information revelation

Auctions with imperfect information

- Bidders typically do not know how other bidders value the object, or not even how they themselves value the object. Thus an auction is typically a **Bayesian** game.
- We can distinguish:
  - **Independent private value auction:** Each bidder's valuation *v* depends only on her own signal and hence she knows her valuation, but has no information about other bidder's valuation other than the initial distribution
    - e.g. a piece of art you buy only because you enjoy it, without considering to resell it.
  - **Common value auction:** Each bidder's valuation is not independent of the other bidder's signal.
  - **Pure common value auction ("mineral rights model"):** Valuation is the same for all bidders (but not known by any of them)
    - e.g. an oil field.

Second-Price Sealed-Bid Auction

- all bidders make simultaneous secret ("sealed") bids
- the highest bid wins the auction and the winner pays the **second highest** bid
- does this make sense, would the auctioneer not prefer the winner to pay his own bid?
- Not necessarily: knowing that they have to pay only the second highest bids, bidders will bid **more aggressively**, so the second highest bid in a 2nd price auction may be higher than the highest bid in a 1st price auction.

Equilibrium for Second-Price Sealed-Bid Auction

Proposition: In the 2nd price sealed bid auction it is a (weakly) dominant strategy to bid $b(v) = v$

SPA induces truth telling!

Proof: Let h = the highest of the other bids

Assume you bid $b < v$.

- If $h < b$, you win and you pay h.
- But by bidding v, you also win and also pay h.
- If $h > v$, then you do not win with either b or v.
- If $b < h < v$, then with b you do not win, but with v you win and make a profit $v - h > 0$.

Thus $b = v$ **weakly dominates** bidding $b < v$.

Assume you bid $b > v$.

- If $h < v$, then you win and pay h both if you bid b or v.
- If $h > b$, then you do not win in either case.
- But if $v < h < b$, you do not win by bidding v but you win by bidding b and pay h.
  - Your profit then is $v - h < 0$.

Thus bidding $b = v$ **weakly dominates** bidding $b > v$

Therefore there is a symmetric equilibrium in weakly dominant strategies where each bidder bids his valuation

Note that this result is **independent** of the **number** of bidders and the **risk preferences** of the bidders

Equilibrium for Second-Price Sealed-Bid Auction

Remark: The proposition also holds for the **English clock auction:**

- There is a public price clock that increases
- Bidders can drop out at any price, but cannot re-enter the auction
- The auction ends when only one bidder is left and the winner pays the price when the second to last bidder quits
- It is weakly dominant to quit when $p = v$

First-price Seale-Bid Auction

- common for auctions run by mail
- all bidders make secret ("sealed") bids x_i
- the highest bid wins the auction and the winner **pays his bid**
- Let h be the highest of the bids of the other bidders
- The expected profit in FPA is $\Pr(b>h)(v - b)$
- Thus there is obviously a **trade-off**: if you raise your bid, you increase the **probability** to win $\Pr(b>h)$, but you reduce the **amount** you win $(v - b)$.
- For example, if as in SPA you bid $b = v$, your profit is 0 for sure
- But if you bid too low, the profit will also be 0

First-price auction with private values

Consider the following special case:

- Let there be **2 bidders** and let the valuations be independently distributed according to a **uniform distribution** on $[0,V]$ and bidders are **risk neutral**

Proposition: There is a symmetric Nash equilibrium where each bidder chooses the bidding function $b(v)=v/2$

Proof: Let w be the other bidder's valuation, h be his equilibrium bidding strategy and g its inverse

- Then $\Pr(b > h(w)) = \Pr(g(b) > w) = g(b)/V$
- Now assume that h is linear $h(w) = aw$, then
- $g(b)/V = b/aV$, so the expected profit is
- $(b/aV)(v - b) = (bv - b^2)/aV$.
- Taking the derivative w.r.t. b yields $(v - 2b)/aV = 0$ if $b=v/2$
- so the profit maximizing bid is $b(v)=v/2$

More generally, we can show:

- In the first-price sealed bid independent private value auction with n bidders and uniformly distributed values, it is a **Bayes-Nash equilibrium** to bid

$$b = v\,(n-1)/n$$

What happens if bidders are **risk averse**?

- expected profit in FPA is $\Pr(b > h)(v - b)$ (1)
- expected utility is $\Pr(b > h)\ u(v - b)$ (2)
- a risk averse bidder has a concave utility function u, so when maximizing (2) the solution will be at a higher $\Pr(b>h)$ and hence at a higher bid than for (1), i.e. the bidder is more aggressive

Revenue Equivalence

Revenue Equivalence Principle: All **standard** auctions (that is those where the highest bidder wins for sure) yield the same (expected) revenue for the auctioneer.

- hence in FPA, bidders shade their bids (compared to SPA) just enough to compensate for the fact that the winner pays the highest instead of the second highest bid.
- Consider again the special case **2 bidders**, **uniform distribution** on $[0,V]$
  - In SPA, the expected revenue is the expected value of the **lower** of the two valuations, this is $V/3$
  - In FPA, the expected revenue is ½ of the expected value of the **higher** of the two valuations, which is $2V/3$, so again $V/3$
- REP also holds for some special auctions like the **all-pay** auction: the highest bidder wins, but **all** pay their bids
  - **Note:** REP refers to expected revenue, for a specific set of valuations, SPA and FPA generally yield different revenues
- REP does not hold for **common value** auctions or if some conditions are not met (e.g. if bidders are **risk-averse**, the first-price auction raises more revenue, WHY?)

The Winner's Curse

Consider a **pure common value** SPA, n bidders

- You receive a **signal** s of the common value V
- What happens if you bid as for private values $b(s) = \mathrm{E}(V \mid s_i)$?
- If all bidders follow this strategy, you win if you have the **highest** signal and pay the second highest signal.
- This is likely higher than V, so you make a loss and incur the **winner's curse**

The winner's curse results from ignoring that **winning the auction is bad news** about the value of the object:

- You win the auction because everybody else believed it was worth **less** than you thought, which most likely means that you got it wrong and pay too much

Equilibrium: $b(s_i) = \mathrm{E}(V \mid s_i = max\{s_k\}, max\{s_k, k \neq i\} = s_i\})$

Note: In equilibrium, the winner's curse does not occur

Experimental Results

Hundreds of (laboratory and field) experiments have been run with different auction formats.

Typical results:

- bidders approach SPA and EA equilibria with some learning
- EA works better than SPA
- there is some **overbidding** in SPA, and more than in EA. Why?
- FPA bids are on average frequently **above** equilibrium
  - Possible explanations: **risk-aversion**, “joy of winning”, bounded rationality
- Thus **revenue equivalence** does frequently not hold, FPA yield more money than SPA
- **Winner’s curse** is frequent

Problem set #07

1. Think about the ultimatum game (we can consider this as a strategic-form game, where the responder decides about his minimal acceptable offer at the same time as the proposer decides about the offer.) Let there be a selfish type of responder (accepts everything) with probability q and a “fair” type of responder who accepts only the equal split (or above) with probability 1-q. Let the proposer be selfish. Determine the Bayes-Nash equilibrium, depending on q.
2. Consider an auction where there are n objects for sale, but each bidder can buy only one object. The n highest bidders win the objects and pay the $(n+1)^{\text{th}}$ highest bid. Show that it is again a dominant strategy to bid your valuation. What happens if individual bidders could buy both units?
3. (How can auctioneers cheat? Is it easier to cheat in FPA or SPA? In which auctions is it easier for bidders to collude?)
4. Osborne Ex 282.2
5. (Osborne Ex 307.1)