EC3224 Autumn Lecture #04
Mixed-Strategy Equilibrium

- Reading
  - Osborne Chapter 4.1 to 4.10
- By the end of this week you should be able to:
  - find a mixed strategy Nash Equilibrium of a game
  - explain why mixed strategies can be important in applications

Example: Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- Matching pennies does not have a Nash equilibrium (in the game with ordinal preferences), i.e. there is no steady state where all players 1 choose the same action and all players 2 choose the same action and nobody wants to change

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- But: we can assume that if a player faces getting 1 with probability *p* and -1 with probability 1-*p,* then the higher *p* the better

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- But: we can assume that if a player faces getting 1 with probability *p* and -1 with probability 1-*p,* then the higher *p* the better
- Now if player 2 chooses H with probability ½, then both H and T yield the same lottery to player 1 (namely getting 1 with probability ½ and -1 with ½)

Example: Matching Pennies

- Thus no matter which strategy player 1 chooses, she has no incentive to change it
- The same holds for player 2: if player 1 chooses H with probability ½ and T with probability ½, both H and T yield the same lottery for him
- Thus player 2 has no incentive to change his strategy
- Therefore, if both players choose H with probability ½ and T with probability ½, no one has an incentive to change
- Thus we have a steady state
- This can be interpreted either as a state where each player randomizes with equal probability between H and T or as game between populations where half of the players choose H and the other half choose T

Generalization: Preliminaries

- Ordinal payoff functions are generally not sufficient in order to express preferences over lotteries
- For example in this case we need to know how much worse is -2 than -1 compared to the difference of 1 and -1

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- Thus we need a **cardinal** payoff function, where differences have meaning
- **von Neumann – Morgenstern (vNM) preferences:** Preferences over lotteries represented by the expected value of a payoff function over the deterministic outcomes ("Bernoulli payoff function")

Definitions

- A **strategic game** (with vNM preferences) consists of
  - a set of players
  - for each player, a set of actions
  - for each player, **preferences** regarding lotteries over action profiles that may be represented by the expected value of a ("Bernoulli") payoff function over action profiles
- **Definition:** A **mixed strategy** of a player is a probability distribution over the player's actions
- can denote this by vector of probabilities $(p_1,\ldots, p_n)$ if A_i is a set of n actions $\{a_1,\ldots, a_n\}$
- Let $\Delta(A_i)$ be the set of mixed strategies over A_i
- **Pure strategy:** mixed strategy that puts probability 1 on a single action, i.e. a deterministic action

Mixed-strategy Nash Equilibrium

- Let α^* be a mixed strategy profile in a strategic game with vNM preferences
- Then α^* is a **(mixed-strategy) Nash equilibrium** if for every player i and every mixed strategy $\alpha_i \in \Delta(A_i)$:

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha_{-i}^*)$$

where $U_i(\alpha)$ is i's expected payoff given α

Proposition: every game with vNM preferences with each player having finitely many available actions has a (mixed-strategy) equilibrium

Note: this does not mean that there is no equilibrium if players have infinitely many actions. There just might be no equilibrium

Best Response Functions

- Best response functions are defined as for deterministic actions
- **Best response function** of i:
- $B_i(\alpha_{-i}) = \{\alpha_i: U_i(\alpha_i,\alpha_{-i}) \geq U_i(\alpha_i',\alpha_{-i})$ for all mixed strategies $\alpha_i' \in \Delta(A_i)\}$
- Set-valued, each member of $B_i(\alpha_{-i})$ is a best response to α_{-i}

Again, we get:

Proposition: α^* is a **Nash equilibrium** if and only if

$$\alpha_i^* \in B_i(\alpha_{-i}^*) \text{ for every } i$$

Best Response Functions for Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- *p:* Probability that 1 chooses Head
- *q:* Probability that 2 chooses Head

Best Response Functions for Matching Pennies

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- *p:* Probability that 1 chooses Head
- *q:* Probability that 2 chooses Head
- $B_1(q) = 1$ if $q > 1/2$; $= 0$ if $q < 1/2$; $=[0,1]$ if $q = 1/2$
- $B_2(p) = 0$ if $p > 1/2$; $= 1$ if $p < 1/2$; $=[0,1]$ if $p = 1/2$

Best Response Functions for Matching Pennies

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- $B_2(p) = 0$ if $p > 1/2$; $= 1$ if $p < 1/2$; $=[0,1]$ if $p = 1/2$
- Equilibrium: $(p, q) = (1/2, 1/2)$

Best response function for Battle of the Sexes

		Player 2	
		Ball	Theatre
Player 1	Ball	2,1	0,0
	Theatre	0,0	1,2

- p: Probability that 1 chooses Ball
- q: Probability that 2 chooses Ball

Best response function for Battle of the Sexes

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- $B_1(q) = 1$ if $q > 1/3$; $= 0$ if $q < 1/3$; $= [0,1]$ if $q = 1/3$
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Best response function for Battle of the Sexes

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		Ball	Theatre
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- $B_1(q) = 1$ if $q > 1/3$; $= 0$ if $q < 1/3$; $= [0,1]$ if $q = 1/3$
- $B_2(p) = 1$ if $p > 2/3$; $= 0$ if $p < 2/3$; $= [0,1]$ if $p = 2/3$
- Equilibria: $(p, q) = (1,1) =$ (Ball, Ball),

  $(p, q) = (0,0) =$ (Theatre, Theatre)

  $(p, q) = (2/3, 1/3)$

Pure-Strategy Equilibria survive allowing for mixing

- In battle of the sexes, the 2 equilibria identified in the game with ordinal preferences survive if we interpret the game as having vNM preferences and allow for mixing
- This is always true and vice versa for pure-strategy equilibria

Pure-Strategy Equilibria survive allowing for mixing

- **Proposition:** any equilibrium of a game with ordinal preference is a pure-strategy equilibrium of the corresponding game with vNM preferences and any pure-strategy equilibrium of a game with vNM preferences is an equilibrium of the corresponding game with ordinal preferences
- So all equilibria of the games we encountered so far stay equilibria if we allow for mixing and interpret the payoff functions as Bernoulli payoff functions

Characterization of mixed-strategy equilibria

Proposition: let α^* be a mixed-strategy equilibrium. Then

- each action a_i that is played by i with positive probability according to α_i^* yields the same expected payoff to i as strategy α_i^*
- every action a_i' that is played by i with probability 0 according to α_i^* yields at most the same expected payoff to i as strategy α_i^*

This is useful for finding mixed-strategy equilibria:

- each player has to be indifferent between all the strategies she plays
- so for a given game and each player i, we can consider subsets of strategies and see whether we find solutions for the **other players' mixed strategies**, such that i is indifferent among all actions in the subset

		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,4
	B	1,5	4,1

No equilibrium where either player plays a pure strategy (easy)

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Is there one where player 1 chooses T and B, M and B or all three actions with positive probability? No: T strictly dominates B, so whatever player 2 does, 1 can increase expected payoff by playing T instead of B (so b = Pr(B) = 0)

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That leaves 1 choosing T and M. This requires for l=Pr(L)

$2l + 5(1 - l) = 3l + 1(1 - l)$, or $5 - 3l = 1 + 2l$ or $l = 4/5$

mixing of 2 between L and R requires for t=Pr(T)

$3t + 2(1 - t) = 4(1 - t)$, or $2 + t = 4 - 4t$ or $t = 2/5$

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mixing of 2 between L and R requires for t=Pr(T)

$3t + 2(1 - t) = 4(1 - t)$, or $2 + t = 4 - 4t$ or $t = 2/5$

Equilibrium: $((t,m,b),(l,1 - l)) = ((2/5,3/5,0),(4/5,1/5))$

Dominated strategies

An action can be dominated by a mixed strategy:

The mixed strategy α_i strictly dominates action a_i' if for all profiles a_{-i} of the other players' actions the expected payoff to α_i is strictly larger than the payoff to a_i', i.e.

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \quad \text{for all } a_{-i}$$

a_i' is then **strictly dominated**

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Example: T is strictly dominated by a mixed strategy that plays M and B both with probability ½.

		Player 2	
		L	R
Player 1	T	1,3	1,0
	M	4,2	0,4
	B	0,5	3,1

Symmetric games

Definitions of symmetric games and symmetric equilibria carry over in obvious way

We had seen that symmetric game with ordinal preferences does not necessarily have a symmetric equilibrium

But a symmetric game with vNM preferences and a finite set of actions **always** has at least one symmetric equilibrium (pure or mixed).

Application: Preparing questions for a seminar

n students, all identical, decide independently

benefit $b > 0$ to each if at least one is prepared, 0 otherwise

cost c with $0 < c < b$ of preparing the question

There are n asymmetric equilibria where exactly one is prepared

Symmetric equilibrium? Has to be mixed.

p: probability to prepare

need U(prepare) = U(not prepare)

$b - c = 0 \Pr(0 \text{ others prepare}) + b \Pr(>0 \text{ others prepare})$

$b - c = b\,(1 - \Pr(0 \text{ others prepare}))$

$c/b = \Pr(0 \text{ others prepare}) = (1 - p)^{n-1}$

$p = 1 - (c/b)^{1/(n-1)}$

Note: p is **decreasing** in n. But also probability that **at least one** is prepared is decreasing in n, because probability that no one is prepared is $(1 - p)\ c/b$, which is increasing in n.

More on mixed-strategy equilibria

It appears a bit odd that players choose their strategies in order to keep the others indifferent.

- It is not claimed that players intentionally do this, but is just an equilibrium requirement
- However, think of repetition of games that have matching pennies structure (e.g. goalie vs. penalty kicker): the players actually would like to keep the other indifferent. **Why?**

More on mixed-strategy equilibria

How could players arrive at their beliefs?

- Deleting dominated strategies
- Best-response dynamics (play best response to previous action of opponent)
- Learning in repeated games
- This illustrates that there are stable mixed equilibria (e.g. in matching pennies) and unstable ones (e.g. in BoS)

Problem set #04
NOTE: I expect that you have tried to solve the exercises *before* the seminar

1. Osborne, Ex 106.2
2. Osborne, Ex 114.2
3. (Osborne, Ex 114.3)
4. (Osborne, Ex 118.2)
5. Osborne, Ex 121.2
6. Osborne, Ex 141.1
7. Osborne, Ex 142.1
8. Show the following: a 2x2 game with two pure **strict** Nash equilibria always has a mixed-strategy equilibrium that is **not** a pure strategy equilibrium.