EC3224 Autumn Lecture #03
Applications of Nash Equilibrium

- Reading
  - Osborne Chapter 3
- By the end of this week you should be able to:
  - apply Nash equilibrium to oligopoly games, voting games and other examples.

Cournot Equilibrium

- The standard ingredients:
  - – n identical firms, with constant marginal costs c
  - – linear inverse demand $P = a - Q$
  - – total quantity $Q = q_1 + \ldots + q_n$
- look for best response function:
  - – Profit for i: $Pq_i - cq_i = (a - Q)q_i - cq_i$
  - – let $Q_{-i} = Q - q_i$
  - – so profit for i: $(a - Q_{-i} - q_i)q_i - cq_i$
  - – taking the derivative: $(a - Q_{-i} - c) - 2q_i$
  - – Best response function $b_i(Q_{-i}) = (a - c - Q_{-i}) / 2$

Cournot Equilibrium

- in equilibrium $q_1 = \ldots = q_n$
  - why? Let $x = q_i - q_k$
  - then $Q_{-i} - Q_{-k} = -x$ and thus $b_i(Q_{-i}) - b_k(Q_{-k}) = x / 2$
  - So $x = q_i - q_k = b_i(Q_{-i}) - b_k(Q_{-k}) = x / 2$
  - Hence $x = 0$
- Put $q_1 = \ldots = q_n$ in best response function

$$q_i = b_i(Q_{-i}) = (a - c - Q_{-i}) / 2$$

$\Rightarrow q_i = (a - c) / 2 - (n - 1)q_i / 2$

$\Rightarrow (n + 1)q_i / 2 = (a - c) / 2$

$\Rightarrow q_i = (a - c) / (n + 1)$

$\Rightarrow Q = n\,(a - c) / (n + 1)$

$\Rightarrow P = a - Q = (a + nc) / (n + 1)$

- Cournot-game is example for game with externalities

Bertrand Equilibrium

- You should remember this from Micro.
- We will discuss some special cases in the seminar
- Read the chapter in Osborne to refresh your memory.

Applications to Politics

- Hotelling’s model of electoral competition:
  - one-dimensional policy space
  - continuum of voters with preferred policy x
  - vote for candidate closest to x
  - median preferred policy is m
  - players: candidates
  - actions: policies
  - preferences: prefer winning alone over tied win, the fewer with whom to tie the better and tied win is better than losing

Equilibrium in Hotelling's model with 2 candidates

- Best response function:
  - if $x_i < x_k$, then i gets all voters with $x < (x_i + x_k)/2$
  - so i wins if x_i is closer to m than x_k
  - Thus $B_i = \{x_i : |x_i - m| < |x_k - m| \}$, for $x_k \neq m$

    $= \{m\}$, for $x_k = m$
  - Only intersection: (m,m)
  - Can also argue directly why this is the unique equilibrium
  - "minimum differentiation" result
  - also applicable to product differentiation

Equilibrium in Hotelling's model with 3 candidates

- First case: 3 candidates are in the race (no decision regarding entry), distribution of voters has no mass points (more specifically, what we need is mass at *m* is < 1/3)
  - Consider possible equilibria

1. $x_i = x_j = x_k$: each candidate gets 1/3 of votes. There are >1/3 of voters to left or right, each candidate wants to deviate
2. $x_i = x_j \neq x_k$ and all candidates tie or *i* and *j* tie: no equilibrium: there are more than 1/3 of voters left or right of x_i and *k* could hence win by moving just next to *i* and *j*
3. $x_i < x_j < x_k$ and two or three candidates tie: no equilibrium: one of those who tie is *i* or *k* (or both), who can then increase her share and win by moving closer to *j*.

Equilibrium in Hotelling's model with 3 candidates

First case, possible equilibria continued:

4. $x_i < x_j < x_k$ and *j* wins: no equilibrium: at least one of *i* and *k* can increase the share and win by moving closer to *j* or tie with *j* by moving to x_j. (because the share of at least one of *i* and *k* must be smaller than 1/3)
5. That leaves $x_i \leq x_j < x_k$ and *k* winning or $x_i < x_j \leq x_k$:and *i* winning. Indeed, we have such equilibria, e.g. if voters are uniformly distributed on [0,1]: $x_i = x_j = ¼$ and $x_k = ¾$. *k* wins. If *i* deviates to x_i'<1/2, then *k* still wins. If *i* deviates to x_i'>1/2, then *j* wins. If *i* deviates to x_i'=1/2, then *j* and *k* tie. And the same holds for *j*.

NOTE: If candidates care not only about winning, but also about there share of votes, there is no equilibrium in pure strategies, because, for $x_i \leq x_j < x_k$, *k* can increase the vote share by moving to the left.

Equilibrium in Hotelling's model with 3 candidates

- Second case: candidates can decide whether to enter, prefer to stay out over lose, but prefer tying for victory over staying out
  - previous analysis shows in any equilibrium with 3 candidates running, there is exactly one winner. But then the two losers would prefer to stay out, so there cannot be an equilibrium where 3 candidates enter.
  - equilibrium where 2 candidates enter?
  - Now this must be $(x_i, x_j) = (m, m)$, but then k can enter at $x_k = m + \varepsilon$ or $x_k = m - \varepsilon$ and win
  - equilibrium where 1 candidate enters?
  - No, if $x_i = m$, second can enter and tie, otherwise second can enter and win

– No (pure-strategy) equilibrium

The war of attrition

- Two players fight, the one who gives in later wins, they value winning with v_1, v_2, cost: time to end
  - actions: decide when to stop t_i
  - Payoff: $u_i(t_i, t_k) = v_i - t_k$ if $t_i > t_k$
    $= -t_i$ if $t_i < t_k$
    $= v_i/2 - t_i$ if $t_i = t_k$
  - Best response function:
  - $B_i(t_k) = \{t_i: t_i > t_k\}$ if $t_k < v_i$
    $= \{0\}$ if $t_k > v_i$
    $= \{t_i: t_i = 0 \text{ or } t_i > t_k\}$ if $t_k = v_i$
  - Equilibria: (t_1, t_2) with either
    $t_1 = 0$ and $t_2 \geq v_1$ or $t_2 = 0$ and $t_1 \geq v_2$
  - No symmetric equilibrium (in pure strategies)

The war of attrition

- Now consider alternative set-up, where the costs equal own action (e.g. expenditure on arms)
  - actions: decide when to stop t_i
  - Payoff: $u_i(t_i, t_k) = v_i - t_i$ if $t_i > t_k$
    $= - t_i$ if $t_i < t_k$
    $= v_i /2 - t_i$ if $t_i = t_k$
  - Best response function (if there is a smallest cost unit ε):
  - $B_i(t_k) = \{t_k + \varepsilon\}$ if $t_k < v_i$
    $= \{0\}$ if $t_k \geq v_i$
  - No equilibrium (in pure strategies)

Problem set #03
NOTE: I expect that you have tried to solve the exercises *before* the seminar

1. Osborne, Ex 59.2
2. (Osborne, Ex 62.1)
3. Osborne, Ex 69.1
4. Osborne, Ex 74.2
5. (Osborne, Ex 80.2)