EC3224 Autumn Lecture #02
Nash Equilibrium

- Reading
  - Osborne Chapters 2.6-2.10, (12)
- By the end of this week you should be able to:
  - define Nash equilibrium and explain several different motivations for it.
  - find the Nash Equilibrium in simple games.
  - identify dominated strategies and solve games through iterative dominance.

Nash Equilibrium - Motivation

- We wish to find **plausible** outcomes in games
  - for example, in order to predict the outcome
- What is a plausible outcome? Possible criteria:
  - No player is **surprised** by the outcome
  - No player acts against her best **interest** given her **expectations**
  - Action profile would be **repeated** if the same people play again, i.e. nobody would want to change his action
    - "no regret"
  - Profile forms a **steady state** if we randomly rematch players drawn from large populations
    - Avoiding people in the street

Nash Equilibrium - Definition

- In a **Nash equilibrium** each player chooses according to **rational choice** given her beliefs about other players' actions and all players' **beliefs** are **correct** (consistent)
- Correct beliefs can be justified by
  - experience – players interact repeatedly with different partners and hence get to know how the **typical** player decides (even though he is not informed about the specific partner in this interaction)
  - logical reasoning of what the other player might plausibly do
- since beliefs are correct, they must be shared
  - two players have the same belief about the actions of a third

Nash Equilibrium – Formal Definition

- Let A_i be the set of actions available for player i
- $a = (a_1, a_2, \ldots, a_i, \ldots)$ be an **action profile**
- write (a_i', a_{-i}) if i chooses a_i', others play according to a
- Then a^* is a **Nash equilibrium** (of a strategic game with ordinal preferences) if for every player i and every action all $a_i \in A_i$:

  $$u_i(a^*) \geq u_i(a_i, a_{-i}^*)$$

  where u_i is the payoff function representing the preferences of player i
- This means given all players follow a^*, no **individual** player would want to deviate
- they could, however, be jointly better off

Example 1: the Prisoner's Dilemma

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	2,2	0,3
	D(efect)	3,0	1,1

- The unique Nash equilibrium is (D,D)
- For every other profile, at least one player wants to deviate
- It is actually irrelevant here what players believe, they prefer D anyway.
- Consider an experiment where players are paid as in the table above. If they do not choose D, does this reject Nash equilibrium?

Example 2: the "Battle of the Sexes"

		Player 2	
		Ball	Theatre
Player 1	Ball	2,1	0,0
	Theatre	0,0	1,2

- There are two Nash equilibria: (Ball, Ball) and (Theatre, Theatre)
- Which one to choose?

Variant of "Battle of the Sexes"

		Player 2	
		Ball	Theatre
Player 1	Ball	2,2	0,0
	Theatre	0,0	1,1

- There are again two Nash equilibria: (Ball, Ball) and (Theatre, Theatre)
- But now the choice seems easy
- Ball equilibrium is "focal point" (Schelling)

Example 3: Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- There is no Nash equilibrium (of the game with ordinal preferences)
- Once we extend the notion of Nash equilibrium, we will find one

Example 4: “Stag-Hunt”

		Player 2	
		Stag	Hare
Player 1	Stag	2,2	0,1
	Hare	1,0	1,1

- There are two equilibria:
- (Stag, Stag) and (Hare, Hare)
- Which equilibrium to choose? Stag could be “focal”
- If there are more players, there are still only two equilibria:
  - either all choose Stag
  - or all choose Hare
- Stag equilibrium might still be focal, but is it plausible?

Strict Nash Equilibrium

- In cases above, each player was always strictly better off than if she deviated
- This is not required by Nash equilibrium
- Example?
- If it holds then we talk of a **strict** Nash equilibrium
- An action profile a^* is a **strict Nash equilibrium** (of a strategic game with ordinal preferences) if for every player i and every action $a_i \in A_i$:

$$u_i(a^*) > u_i\,(a_i,\, a_{-i}{}^*)$$

where u_i is the payoff function representing the preferences of player i

Strictly Dominated Strategies

- Finding a Nash equilibrium is sometimes easy because we can exclude **dominated** strategies
- a_i' **strictly dominates** a_i if for all strategy profiles a_{-i} of the other players

$$u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$$

- a_i is then **strictly dominated**
- a strictly dominated strategy **cannot** be chosen in any Nash equilibrium (why?)
- Thus we can start by eliminating strictly dominated strategies
- Example: Prisoner's dilemma, C is strictly dominated

Example: B is dominated

		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,1
	B	1,0	4,1

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		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,1
	~~B~~	~~1,0~~	~~4,1~~

Dominant strategies

- If a strategy strictly dominates **all** other strategies, it is **strictly dominant**
- a_i' is a **strictly dominant strategy** for player i if for all $a_i \neq a_i'$ and all strategy profiles a_{-i} of the other players

$$u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$$

- if a player has a strictly dominant strategy, this **must** be played in a Nash equilibrium (why?)
- Then we can fix a_i' and continue by finding the other players' best responses

Example: L is Strictly Dominant

		Player		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,5	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,0

Example: L is Strictly Dominant

		Player		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,5	5,3	3,1
	Z	4,3	1,1	2,2
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Iterated Elimination of Dominated Strategies

- When searching for Nash equilibria, we can **eliminate strictly dominated strategies**
- What happens then?
- If after eliminating strictly dominated strategies a player has again a strictly dominated strategy, this cannot be played in any Nash equilibrium (why?)
- Continue…
- All Nash equilibria must be among the surviving profiles, but not all surviving profiles are Nash-equilibria

Example: Iterated Elimination of Strictly Dominated Strategies

		Player		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

Example: Iterated Elimination of Strictly Dominated Strategies

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Example: Iterated Elimination of Strictly Dominated Strategies

		Player		
		L	C	~~R~~
Player 1	~~T~~	~~2,3~~	~~2,2~~	~~5,0~~
	Y	3,2	5,3	~~3,1~~
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Example: Iterated Elimination of Strictly Dominated Strategies

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	Y	3,2	5,3	~~3,1~~
	Z	4,3	1,1	~~2,2~~
	~~B~~	~~1,2~~	~~0,1~~	~~4,4~~

Weakly Dominated Strategies

- a_i' **weakly dominates** a_i if for all strategy profiles a_{-i} of the other players

  $$u_i(a_i', a_{-i}) \geq u_i(a_i, a_{-i})$$

  and there is at least one a_{-i}' such that

  $$u_i(a_i', a_{-i}') > u_i(a_i, a_{-i}')$$
- a_i is then **weakly dominated**
- a weakly dominated strategy **can** be chosen in a Nash equilibrium
- while (iterated) elimination of weakly dominated strategies is plausible and sometimes leads to a Nash equilibrium, it is problematic because we may eliminate equilibria and the order of elimination matters

Best Response Functions

- Above, we could find the equilibrium just by going through the few possible action profiles.
- This is getting messy if there are many available actions (in particular if there are infinitely many)
- Consider **best response function** of i:
- $B_i(a_{-i}) = \{a_i \in A_i: u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \in A_i\}$
- Set-valued ("correspondence"), each member of $B_i(a_{-i})$ is a best response to a_{-i}

Best Response Functions

Proposition: a^* is a **Nash equilibrium** if and only if

$$a_i^* \in B_i(a_{-i}^*) \text{ for every } i \qquad (1)$$

If all players have unique best responses for each combination of others' actions, i.e.

$$B_i(a_{-i}) = \{b_i(a_{-i})\} \text{ then (1) becomes}$$

$$a_i^* = b_i(a_{-i}^*) \text{ for every } i \qquad (2)$$

To find Nash equilibrium:

- find best response functions for each player
- find a^* that satisfies (1) (or (2) if best responses have only one value)

Best Response Functions - Example

Ex 38.2 (extended): two players divide £10. Players make demands a_1, a_2. If $a_1 + a_2 \leq 10$, they get a_1, a_2. If $a_1 + a_2 > 10$ and $a_i \leq 5$, i gets a_i and j gets $10 - a_i$. If $a_1 > 5$ and $a_2 > 5$, they both get 5.

Best Response Functions - Example

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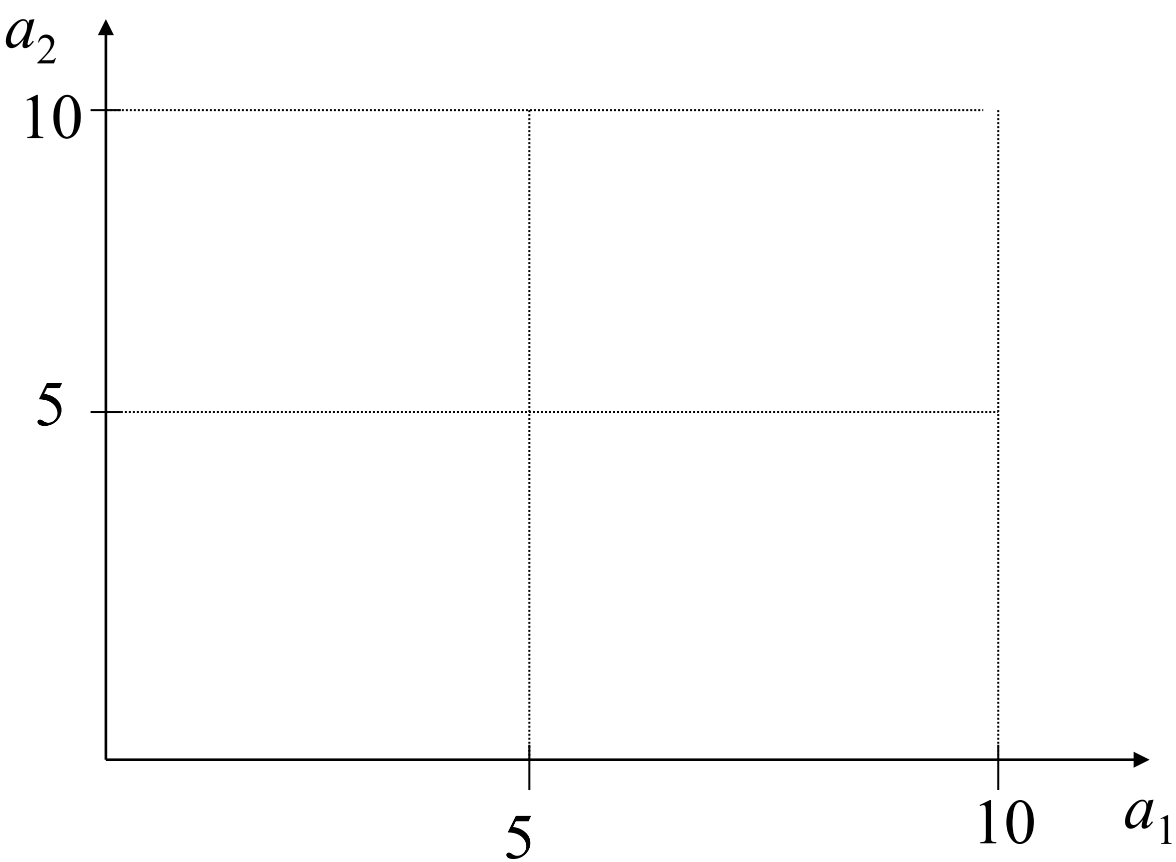

Best Response Functions - Example

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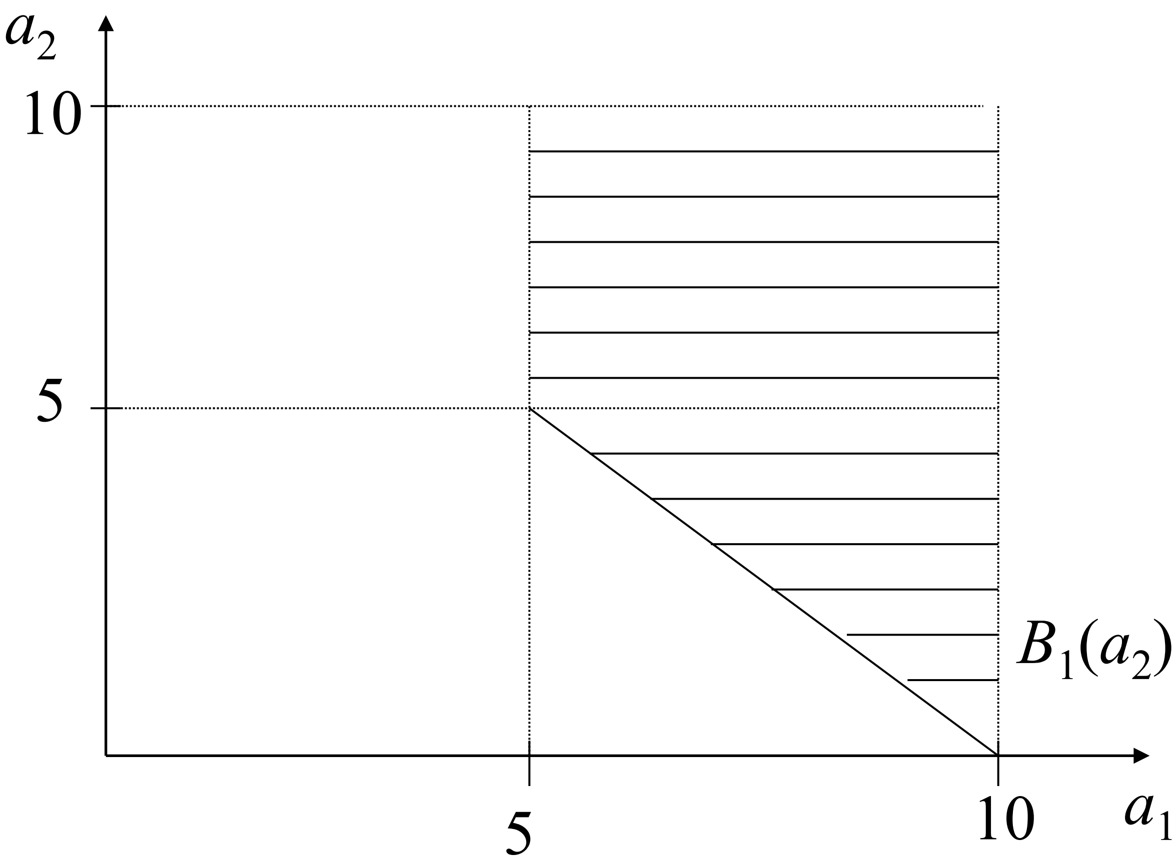

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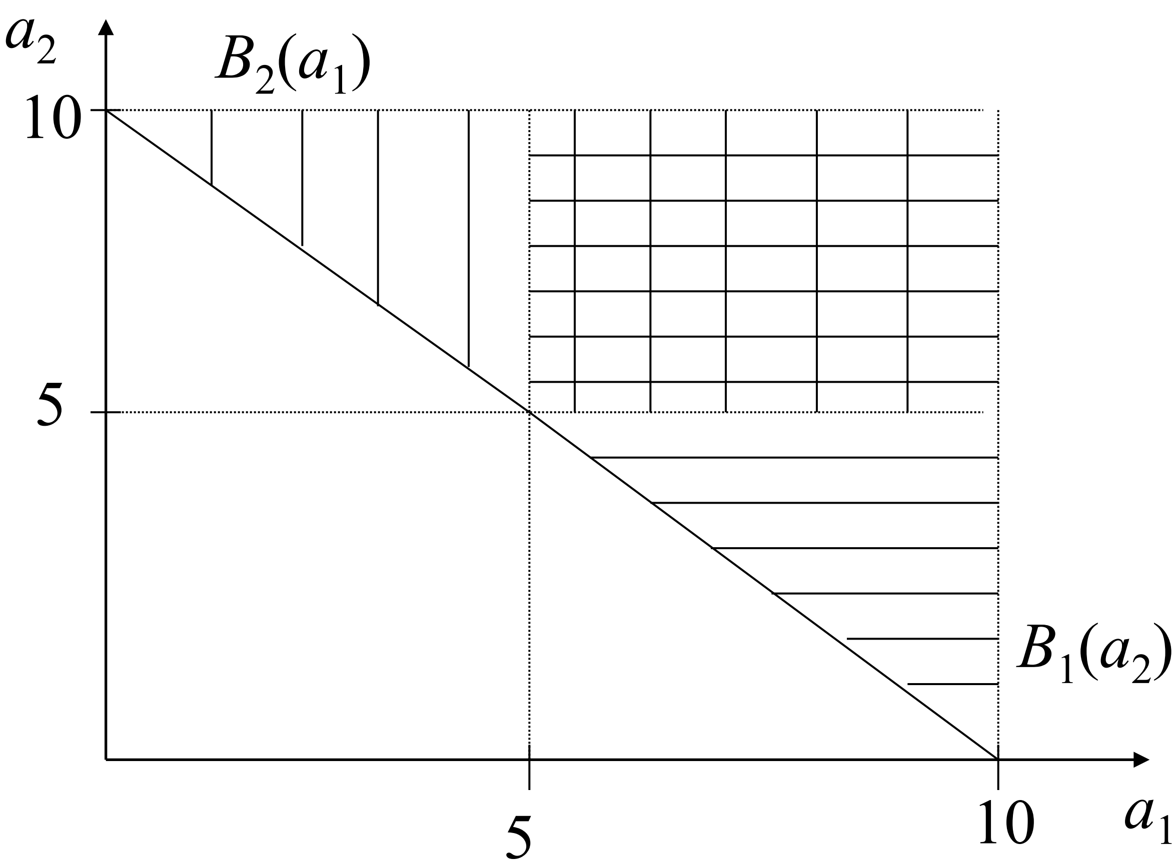

Best Response Functions - Example

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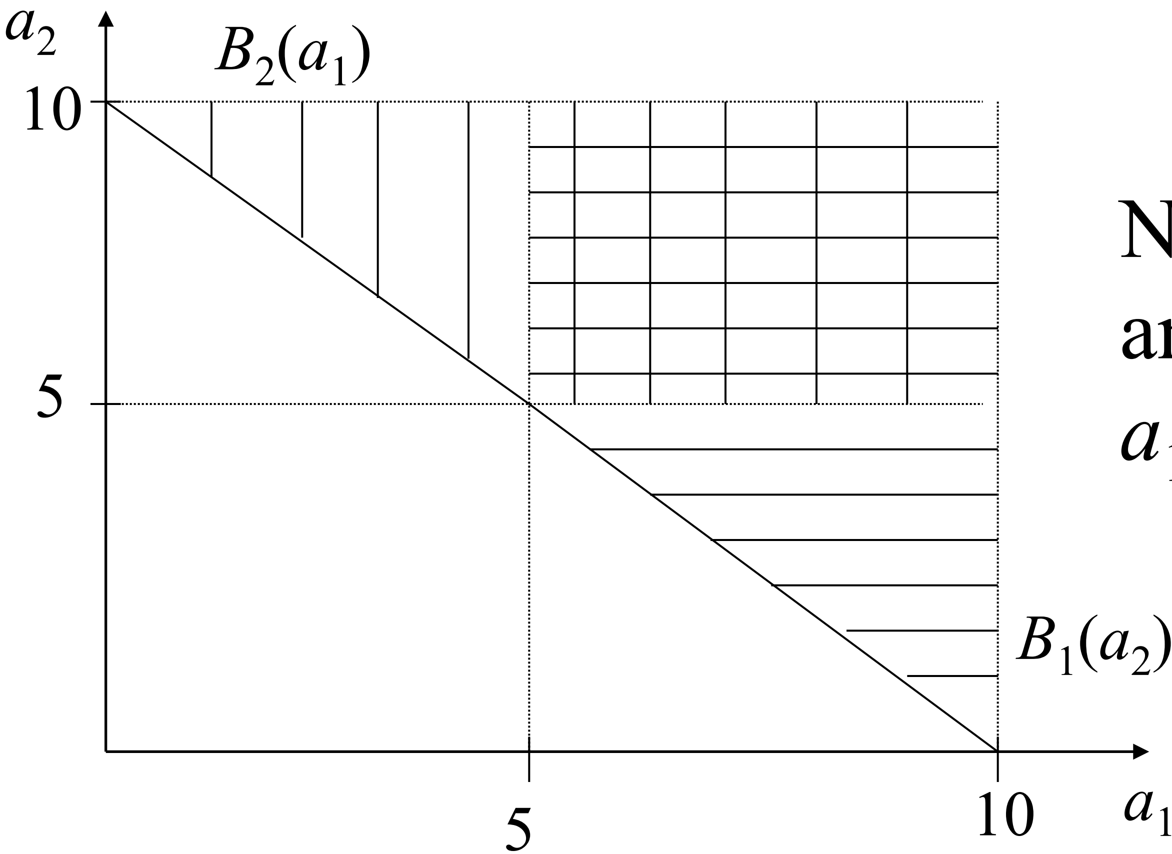

Nash-equilibria: any (a_1, a_2) with $a_1 \geq 5$, $a_2 \geq 5$

Symmetric Games

- Nash equilibrium of *n*-player games corresponds to steady state in game between randomly drawn members of *n* populations
- What if there is no difference between players except the label?
- i.e. players are from a **single** population?
- **Symmetric game:** $A_1 = A_2$ and $u_1(a_1, a_2) = u_2(a_2, a_1)$
- **Example:** Prisoner's dilemma, stag-hunt, but not BoS or matching pennies
- if players are from the same population, in steady state all choose the same action
- a^* is a symmetric **Nash equilibrium** (of a strategic game with ordinal preferences) if it is a Nash equilibrium and a_i^* is the same for all i

Symmetric Games

- symmetric games do not need to have symmetric equilibria

	X	Y
X	0,0	2,1
Y	1,2	1,1

- NE: (X,Y) and (Y,X).
- Could we expect these as steady states in a single population?

Problem set #02
NOTE: I expect that you have tried to solve the exercises *before* the seminar

1. Find the Nash equilibria in the bank-run game. Discuss why one equilibrium is becoming more plausible if the number of players increases
2. Osborne, Ex 27.1
3. Osborne, Ex 27.2
4. (Osborne, Ex 31.1. Is any of the equilibria focal? What do you think happens if this game is played in an experiment in a group of 2 people? What if it is played in a group of 9 people?)
5. Osborne, Ex 34.2
6. (Osborne, Ex 34.3)
7. Osborne, Ex 42.2
8. Osborne, Ex 48.1
9. Write down a 3x3 game matrix where a Nash-equilibrium is eliminated by iterated elimination of weakly dominated strategies and this depends on the order of elimination