Product Differentiation: Exercises Part 1

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Problem 1

Consider Hotelling's linear city with endogenous prices and exogenous and locations. Suppose, however, that there is only one firm, and that this monopolist is (exogenously) located at the left end point of the interval ($y_1 = 0$). Assume that the consumers' transportation costs are quadratic, $k(x-y)^2$, where $k > 0$ is a parameter, x is the location of the consumer, and y is the location of the firm from which she buys. Also, allow for the possibility that some consumers may prefer not to buy at all (which yields zero utility). Solve for the optimal monopoly price in this model assuming zero production costs.

Solution Problem 1

We thus have only one firm, located at $y_1 = 0$. Consumers are assumed to be uniformly distributed over the unit interval. The timing of the game is

1. The firm chooses a price.
2. Each consumer chooses whether to buy or not.

Since the game is dynamic we will use backward induction to solve for a subgame perfect equilibrium.

In the first step hence take the firm's price p_1 as given an consider the problem faced by the consumers. The problem faced by any given consumer is whether or not to buy from the firm. When we solve this problem for all consumers, and for a generic price p_1, we thus trace out the firms demand function.

Since the consumers are identical except for their locations, and moreover, the transportation cost increases in distance from the producer, there will be some consumer (location), denoted $\overline{x}$, who is indifferent between buying and not buying. Everyone to the left of $\overline{x}$ will buy, while everyone to the right of $\overline{x}$ will not. When solving for $\overline{x}$ we have to be a bit care, however, since it may be that we have a corner solution where either all consumers buy ($\overline{x} > 1$) or no consumers buy.

The utility to consumer x from buying the good is

$$U(\theta^*, 0) = V - kx^2 - p_1.$$

In contrast, a consumer with location x gets zero utility if not buying. Hence for the indifferent consumer $\overline{x}$ we have that

$$V - k\overline{x}^2 - p_1 = 0.$$

Solving for $\overline{x}$, we have

$$\overline{x} = \sqrt{\frac{V - p_1}{k}}.$$

However, as noted above, we need to be a bit careful since this expression is only valid when the resulting cutoff point is in the unit interval. It is thus only valid for prices $V \geq p_1 \geq V - k$. To see this, note that if $p_1 \geq V$, then not even the consumer at that is at the firm's location, i.e. $x = 0$, will want to buy; hence at such a high price demand is zero. Conversely, if the price is $p_1 < V - k$, then even the most distant consumer (at $x = 1$) will strictly prefer to buy, implying that demand is equal to one.

Since the consumers are uniformly distributed over the unit interval, the demand for the firm's product is thus

$$D_1(p_1) = \overline{x} = \sqrt{\frac{V - p_1}{k}}.$$

Having derived this demand function, we can now consider the firm's price setting problem. Since the cost of production is zero, the firm's profits are

$$\pi_1(p_1) = p_1 D_1(p_1) = p_1 \sqrt{\frac{V - p_1}{k}}.$$

The first order condition satisfied by the optimal price is

$$\left(\frac{V - p_1}{k}\right)^{\frac{1}{2}} - \frac{p_1}{2k}\left(\frac{V - p_1}{k}\right)^{-\frac{1}{2}} = 0.$$

Solving the first order condition yields

$$\begin{aligned}
\left(\frac{V-p_1}{k}\right)^{\frac{1}{2}} &= \frac{p_1}{2k}\left(\frac{V-p_1}{k}\right)^{-\frac{1}{2}} \Leftrightarrow \\
\left(\frac{V-p_1}{k}\right) &= \frac{p_1}{2k} \Leftrightarrow \\
2\left(V-p_1\right) &= p_1 \Leftrightarrow \\
p_1^* &= \frac{2}{3}V
\end{aligned}$$

Plugging the optimal price back into the expression for the profits yields that

$$\begin{aligned}
\pi_1^* &= p_1^*\left(\frac{V-p_1^*}{k}\right)^{\frac{1}{2}} = \frac{2}{3}V\left(\frac{V}{3k}\right)^{\frac{1}{2}} \\
&= \frac{2V^{\frac{3}{2}}}{3\left(3k\right)^{\frac{1}{2}}}
\end{aligned}$$

Given that $V > 0$ and $k > 0$ this expression is strictly positive.

However, this optimal profit was derived under the assumption that the indifferent type $\overline{x}$ was interior, i.e. within the unit interval, and hence only valid if $V - k \leq p_1^* \leq V$. Clearly $p_1^* < V$ (since it is a fraction of V). This tells us that the firm is, trivially, better off setting a price that induces some consumers to buy, i.e. $p_1 \geq V$ would not be optimal. However, we also need to consider the possibility that it might be optimal to set a price that is low enough that all consumers buy.

What are the highest possibly profits for the firm when all consumers buy? Clearly, this is achieved by setting $\widehat{p}_1 = V - k$ (since this is the highest price at which all consumers buy). Profits are then

$$\widehat{\pi}_1 = \widehat{p}_1 \cdot 1 = V - k.$$

When does inducing all consumers to buy generate the highest profits? It will do so when the unconstrained price p_1^* that we derived above is lower than $\widehat{p}_1$, i.e. when

$$p_1^* = \frac{2}{3}V < V - k = \widehat{p}_1$$

or, equivalently, when

$$k < \frac{V}{3}.$$

Another way of seeing this it to think about the price setting problem as a constrained optimization problem,

$$\max_{V-k\leq p_1\leq V} \pi_1(p_1)$$

In this formulation, and knowing that the profit function $\pi_1(p_1)$ is a concave function, checking whether setting the price at the lower bound is optimal involves checking whether the derivative of $\pi_1(p_1)$ with respect to p_1, when evaluated at $p_1 = V - k$, is positive or negative. If it is positive, then the optimal price is larger than $V - k$, while if it is negative, the optimal price is $V - k$. Recall that the derivative is

$$\pi_1'(p_1) = \left(\frac{V-p_1}{k}\right)^{\frac{1}{2}} - \frac{p_1}{2k}\left(\frac{V-p_1}{k}\right)^{-\frac{1}{2}}$$

Hence evaluating at $p_1 = V - k$ yields

$$\begin{aligned}\pi_1'(p_1 = V-k) &= \left(\frac{V-(V-k)}{k}\right)^{\frac{1}{2}} - \frac{(V-k)}{2k}\left(\frac{V-(V-k)}{k}\right)^{-\frac{1}{2}} \\ &= 1 - \frac{(V-k)}{2k}\end{aligned}$$

The derivate of profits, evaluated at the lower bound of prices, is thus positive if $k > \frac{V}{3}$ and negative at $k < \frac{V}{3}$.

Hence we conclude that, if $k > V/3$ then, at the optimum, the monopolist does not sell to all consumers and sets the optimal price $p_1^* = (2/3)V$, while if $k \leq V/3$, it sells to all consumers and sets the optimal price $p_1 = V - k$.

Problem 2

Consider a Hotelling model with two firms: firm 1 is located at $y_1 = 0$, and firm 2 is located at $y_2 = 1$. Consumers are uniformly distributed along the interval $[0, 1]$. Each consumer wishes to buy at most one unit. The utility of a consumer located at x is

$$V_1 - p_1 - kx^2$$

if he buys from firm 1,

$$V_2 - p_2 - k(1-x)^2$$

if he buys from firm 2, and 0 if he buys from neither firm. V_i represents the "qualities" of the products offered by firm i, while p_i is the price set by firm i. k is a positive constant. For simplicity, assume that the two firms have zero production costs and that they compete by simultaneously setting prices.

(a) Given p_1 and p_2, compute the location of the consumer who is just indifferent between the two firms (suppose that the market is covered). Explain the intuition of the expression you got.

(b) Given your answer in (a), write the profit maximization problem of each firm. Solve the problem and derive the best-response function of each firm. Show the two best-response functions in a graph that has p_1 on the horizontal axis and p_2 on the vertical axis, assuming $V_1 = V_2$ and $k = 1$. Solve for the equilibrium set of prices given that $V_1 = V_2$.

(c) Suppose that firm 1 increases V_1 by an amount $a > 0$ by investing in quality (so that $V_1 = V_2 + a$). What is the resulting change in the best response functions of the two firms? Illustrate your answer with a figure and explain the intuition for the resulting changes. Compute the equilibrium prices after the increase in quality by firm 1.

(d) Is the "strategic effect" of the increase in V_1 beneficial or harmful for firm 1? Would firm 1 be more inclined or less inclined to invest relative to the case where it does not engage in price competition with firm 2? Explain your answer. (See CW, p. 532 for a general discussion of strategic effects.)

Solution Problem 2

(a) The location of the indifferent consumer, denoted $\overline{x}$, is given by the solution to the following equation,

$$V_1 - p_1 - k\overline{x}^2 = V_2 - p_2 - k(1-\overline{x})^2$$

or

$$\begin{aligned} k(1-\overline{x})^2 - k\overline{x}^2 &= V_2 - V_1 + p_1 - p_2 \text{ (rearrange)} \\ k - 2k\overline{x} &= V_2 - V_1 + p_1 - p_2 \text{ (simplify l.h.s)} \\ 2k\overline{x} - k &= V_1 - V_2 - p_1 + p_2 \text{ (multiplying by -1)} \end{aligned}$$

or, finally solving,

$$\overline{x} = \frac{1}{2} + \frac{V_1 - V_2 - p_1 + p_2}{2k}$$

This expression is the demand for firm 1 while $1-\overline{x}$ is the demand for firm 2. If firms 1 and 2 offers packages that have the same quality and price, then the each firm captures half of the market. Firm i can capture more than half of the market by either offering a lower price or by offering a higher quality.

(b) Define the demand function for each firm

$$D_1(p_1, p_2; V_1, V_2) = \overline{x} = \frac{1}{2} + \frac{V_1 - V_2 - p_1 + p_2}{2k}$$

and

$$D_2(p_1, p_2; V_1, V_2) = 1 - \overline{x} = \frac{1}{2} + \frac{V_2 - V_1 - p_2 + p_1}{2k}$$

Since production costs are zero, the profits for firm i is simply its revenue $p_i D_i$. Hence, in a Nash price-setting equilibrium, firm 1 chooses p_1, given p_2, so as to maximize

$$p_1 D_1 = p_1 \left(\frac{1}{2} + \frac{V_1 - V_2 - p_1 + p_2}{2k} \right).$$

The first order condition for this problem is

$$\begin{aligned} D_1 + p_1 \frac{\partial D_1}{\partial p_1} &= 0 \\ \left(\frac{1}{2} + \frac{V_1 - V_2 - p_1 + p_2}{2k} \right) - p_1 \frac{1}{2k} &= 0 \\ \frac{1}{2} + \frac{(V_1 - V_2)}{2k} + \frac{p_2}{2k} - \frac{p_1}{k} &= 0 \end{aligned}$$

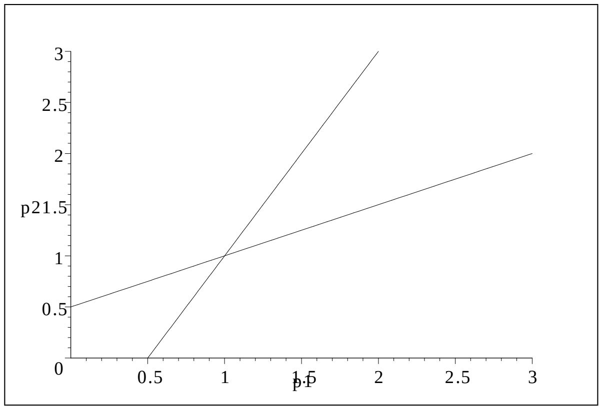
Hence, solving for the best response p_1, yields

$$p_1(p_2; V_1, V_2) = \frac{k}{2} + \frac{(V_1 - V_2)}{2} + \frac{p_2}{2}$$

By a similar calculation, we obtain that the best price-response by firm 2 is

$$p_2(p_1; V_1, V_2) = \frac{k}{2} + \frac{(V_2 - V_1)}{2} + \frac{p_1}{2}$$

Inspecting the best-response function $p_1(p_2; V_1, V_2)$, we see firm 1's chosen price increases by £0.5 for every £1 increase in p_2. Hence in terms of a graph that has p_1 and p_2 on the horizontal and vertical axis respectively, the slope of $p_1(p_2; V_1, V_2)$ is 2. By the same argument, the slope of $p_2(p_2; V_1, V_2)$ is 1/2. The following graph illustrates the case where $k = 1$ and $V_1 = V_2$.



With $V_1 = V_2$ the best response functions simplify to

$$p_1(p_2; V_1, V_2) = \frac{k}{2} + \frac{p_2}{2}$$
$$p_2(p_1; V_1, V_2) = \frac{k}{2} + \frac{p_1}{2}$$

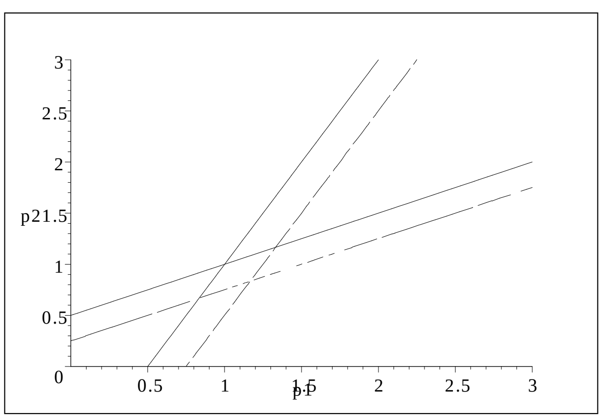
and it is easy to see that the unique price equilibrium is the symmetric outcome

$$p_1 = p_2 = k.$$

(c) Suppose that firm 1 increases the quality of its product so that $V_1 = V_2 + a$, for some $a > 0$. The best-response functions are now

$$p_1(p_2; V_1, V_2) = \frac{k}{2} + \frac{a}{2} + \frac{p_2}{2}$$
$$p_2(p_1; V_1, V_2) = \frac{k}{2} - \frac{a}{2} + \frac{p_1}{2}$$

Hence, relative to the initial situation, the best price-response by firm 1 has increased by $a/2$ while that of firm 2 has decreased by $a/2$. The following figure illustrate the case where $a = 1/2$.



In intuitive terms, when firm 1 raises V_1 than its best-response function shifts outward because it is optimal for firm 1 to raise p_1 for each value of p_2. This is since firm 1's product is now of superior quality, so firm 1 can exploit the higher willingness of its customers to pay by charging them higher prices.

The best-response function of firm 2, however, decreases in the sense that now firm 2 would like to set a lower price p_2 for each value of p_1. The reason for that is that the increase in V_1 has shifted some customers away from firm 2 to firm 1 due to the increase in the quality by firm 1. Hence, in order to regain some customers, firm 2 will want to drop its price.

The new equilibrium entails a lower price by firm. With $a = 1/2$, the new (asymmetric) price equilibrium is the pair (p_1, p_2) the solve the two-equation system

$$p_1 = \frac{k}{2} + \frac{a}{2} + \frac{p_2}{2}$$

$$p_2 = \frac{k}{2} - \frac{a}{2} + \frac{p_1}{2}$$

Using the first equation to substitute for p_1 in the second yields the following equation in p_2

$$p_2 = \frac{k}{2} - \frac{a}{2} + \frac{1}{2}\left(\frac{k}{2} + \frac{a}{2} + \frac{p_2}{2}\right)$$

Simplifying (collecting constants) and solving yields $p_2 = k - a/3$. Plugging this into the first equation then yields

$$p_1 = \frac{k}{2} + \frac{a}{2} + \frac{1}{2}\left(k - \frac{a}{3}\right) = k + \frac{a}{3}$$

Hence with the quality gap $V_1 - V_2 = a$, the new equilibrium prices are

$$(p_1, p_2) = \left(k + \frac{a}{3}, k - \frac{a}{3}\right)$$

(which obviously generalize the case where $a = 0$).

(d) Recall from CW the notion of a strategic effect. Consider two firms 1 and 2 and a two-stage situation; in the first stage firm 1 can make an investment a, and in the second stage the two firms compete e.g. in prices (or alternatively in quantities). Profits for firm 1 can generally be written as $\pi_1 = \pi_1(p_1, p_2, a)$. Using that the outcome of the price competition will depend on the level of the investment, i.e. $p_i = p_i(a)$, the incentives to invest can be written as

$$\frac{\partial \pi_1}{\partial a} + \frac{\partial \pi_1}{\partial p_1}\frac{\partial p_1}{\partial a} + \frac{\partial \pi_1}{\partial p_2}\frac{\partial p_2}{\partial a}$$

The first term is the direct effect – in the current example the increase in quality by firm 1 attracts customers and hence directly increase profits. The second term involves the expression $\partial \pi_1 / \partial p_1$ which is the impact on profits of a marginal change in the price; the condition for a (second-stage) Nash price-setting equilibrium ensures that this term is zero. The third term is the *strategic effect.* Firm 1 recognizes that this will have an effect on its own prices and takes this into account when making the investment decision.

In this case firm 1 recognizes that investing in quality generates a negative strategic effect since firm 2 becomes more aggressive in its price-setting which hurts firm 1. In other words, while firm 1 enjoys a higher quality and hence a higher willingness of its consumers to pay for its product, it also finds itself competing against a more aggressive rival who cuts its price in order to "lure" customers away from firm 1. This harm then must be taken into account when considering the quality increasing investment.