Collusion: Exercises Part 2

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Problem 1 (Problem 8 in Chapter 10 in Church and Ware: Price Setting Collusion with Differentiated Products).

An industry consists of two firms. The demand function for the product of firm i is

$$q_i\left(p_i, p_j\right) = a - bp_i + cp_j.$$

The marginal cost of each firm is assumed to be zero.
(a) Find the critical value of the discount factor that supports joint profit maximization with grim punishment strategies. [Hint: Let $r = c/b$, where $0 < r < 1$.]
(b) How does the critical value of the discount factor depend on the degree of product differentiation r? What does $r = 1$ imply about the relationship between the two goods? $r = 0$?

Solution Problem 1

This is a problem based on price competition. Hence unlike the example in the previous seminar, we will be considering Bertrand price competition as the non-cooperative Nash equilibrium which will serve as the continuation equilibrium in case cooperation breaks down. For the rest, the analysis is very similar to that in the Cournot context.

Let δ denote the discount factor, $0 < \delta < 1$. We start by characterizing the collusive agreement, that is the output/prices that maximize the joint profits of the two firms. Note that profits for firm i, written as a function of prices, is

$$\pi_i = \left(a - bp_i + cp_j\right) p_i.$$

Hence joint profits are

$$\Pi = \pi_1 + \pi_2 = \left(a - bp_1 + cp_2\right) p_1 + \left(a - bp_2 + cp_1\right) p_2$$

The monopoly/cartel prices, which we can denote (p_1^m, p_2^m), maximize the joint profits Π. The first order conditions for a maximum of Π are

$$-bp_1 + (a - bp_1 + cp_2) + cp_2 = 0$$

$$-bp_2 + (a - bp_2 + cp_1) + cp_1 = 0$$

Since the firms are identical, it is natural to look for a symmetric optimum $p_1^m = p_2^m = p^m$ (indeed, one can easily show that this is the only possibility). With symmetry the two equations are identical and each imply that

$$-bp^m + (a - bp^m + cp^m) + cp^m = 0$$

which, upon solving, yields that

$$p^m = \frac{a}{2(b-c)}.$$

In the cartel outcome, the profit of each firm is

$$\begin{aligned}
\pi^m &= (a - bp^m + cp^m)\, p^m \\
&= \left[a - b\left(\frac{a}{2(b-c)}\right) + c\left(\frac{a}{2(b-c)}\right)\right]\frac{a}{2(b-c)} \\
&= \left[a - \frac{a}{2(b-c)}(b-c)\right]\frac{a}{2(b-c)} \\
&= \left[a - \frac{a}{2}\right]\frac{a}{2(b-c)} \\
&= \frac{a^2}{4(b-c)}
\end{aligned}$$

Next we consider the optimal price deviation by firm i from the cartel agreement. Hence imagine that firm i believes that firm j will set the collusive price p^m. If firm i then sets the price p_i its profits will be

$$\pi_i = (a - bp_i + cp^m)\, p_i$$

The optimal deviation, which we can denote p^r, is the price that maximizes this (short-term) profit; the first order condition is

$$-bp^r + (a - bp^r + cp^m) = 0$$

Solving for the optimal deviation yields

$$p^r = \frac{a + cp^m}{2b}$$

But we know that the collusive price was $p^m = a/\left[2\left(b-c\right)\right]$. Hence

$$\begin{aligned}
p^r &= \frac{a + c\left(\frac{a}{2(b-c)}\right)}{2b} \\
&= \frac{\frac{2a(b-c)}{2(b-c)} + \frac{ac}{2(b-c)}}{2b} \\
&= \frac{\frac{1}{2(b-c)}\left(2a\left(b-c\right) + ac\right)}{2b} \\
&= \frac{\left(2a\left(b-c\right) + ac\right)}{4b\left(b-c\right)} \\
&= \frac{a\left(2b-c\right)}{4b\left(b-c\right)}
\end{aligned}$$

The associated (short-run) profit for the deviating firm is

$$\begin{aligned}
\pi^r &= \left(a - bp^r + cp^m\right)p^r \text{ (plug in prices)} \\
&= \left(a - b\frac{a\left(2b-c\right)}{4b\left(b-c\right)} + c\frac{a}{2\left(b-c\right)}\right)\frac{a\left(2b-c\right)}{4b\left(b-c\right)} \text{ (factor } a \text{ and common denom.)} \\
&= a\left(\frac{4\left(b-c\right)}{4\left(b-c\right)} - \frac{\left(2b-c\right)}{4\left(b-c\right)} + \frac{2c}{4\left(b-c\right)}\right)\frac{a\left(2b-c\right)}{4b\left(b-c\right)} \text{ (simplify large paranthesis)} \\
&= a\left(\frac{2b-c}{4\left(b-c\right)}\right)\frac{a\left(2b-c\right)}{4b\left(b-c\right)} \text{ (combine terms)} \\
&= \frac{a^2\left(2b-c\right)^2}{16b\left(b-c\right)^2}.
\end{aligned}$$

So far we have characterized the collusive agreement and the optimal deviation. We next need to characterize the Bertrand price setting equilibrium which will serve as the continuation equilibrium if some firm deviates from the collusive agreement.

In the Bertrand equilibrium firm i sets its price p_i taking the price of the other firm as given; the profits of firm i when it sets price p_i and firm j sets price j are

$$\pi_i = \left(a - bp_i + cp_j\right)p_i$$

The first order condition characterizing firm i's optimal price (response) is hence

$$-bp_i + \left(a - bp_i + cp_j\right) = 0$$

Solving for p_1 in terms of p_2 thus yields

$$p_1(p_2) = \frac{a + cp_2}{2b}$$

and similarly, for firm 2, the best price (response) is

$$p_2(p_1) = \frac{a + cp_1}{2b}.$$

The Bertrand equilibrium will be symmetric, $p_1^b = p_2^b = p^b$, and will hence satisfy

$$p^b = \frac{a + cp^b}{2b}$$

which yields the solution

$$p^b = \frac{a}{2b - c}.$$

A firm's profit in the Bertrand equilibrium:

$$\begin{aligned}
\pi^b &= \left(a - bp^b + cp^b\right) p^b \\
&= \left[a - (b - c) p^b\right] p^b \text{ (Factor in bracket)} \\
&= \left[a - (b - c) \frac{a}{2b - c}\right] \frac{a}{2b - c} \text{ (Substitute for } p^b\text{)} \\
&= a \left[\frac{2b - c}{2b - c} - \frac{(b - c)}{2b - c}\right] \frac{a}{2b - c} \text{ (Factor } a \text{ and set common denom.)} \\
&= a \left[\frac{b}{2b - c}\right] \frac{a}{2b - c} \text{ (Simplify expression in brackets)} \\
&= \frac{ba^2}{(2b - c)^2}.
\end{aligned}$$

We now turn to the main task of characterizing the minimum δ required to sustain collusion as an SPNE with grim trigger strategies. From the previous seminar we know that this requires that

$$\delta \geq \delta^{crit} \equiv \frac{\pi^r - \pi^m}{\pi^r - \pi^b}.$$

It will now be useful to define the relative

$$r \equiv \frac{c}{b}$$

as the relative impact of the own price p_i and the competitor's price on the demand for firm i's output. We assume that $0 < r < 1$. Using this definition, the expression for the profit levels simplify substantially; in particular, by substituting for $c = rb$ we obtain

$$\pi^m = \frac{a^2}{4(b-c)} = \frac{a^2}{4(b-rb)} = \frac{a^2}{4b(1-r)},$$

$$\pi^r = \frac{a^2(2b-c)^2}{16b(b-c)^2} = \frac{a^2(2b-br)^2}{16b(b-br)^2} = \frac{a^2(2-r)^2}{16b(1-r)^2},$$

$$\pi^b = \frac{ba^2}{(2b-c)^2} = \frac{ba^2}{(2b-rb)^2} = \frac{a^2}{b(2-r)^2}.$$

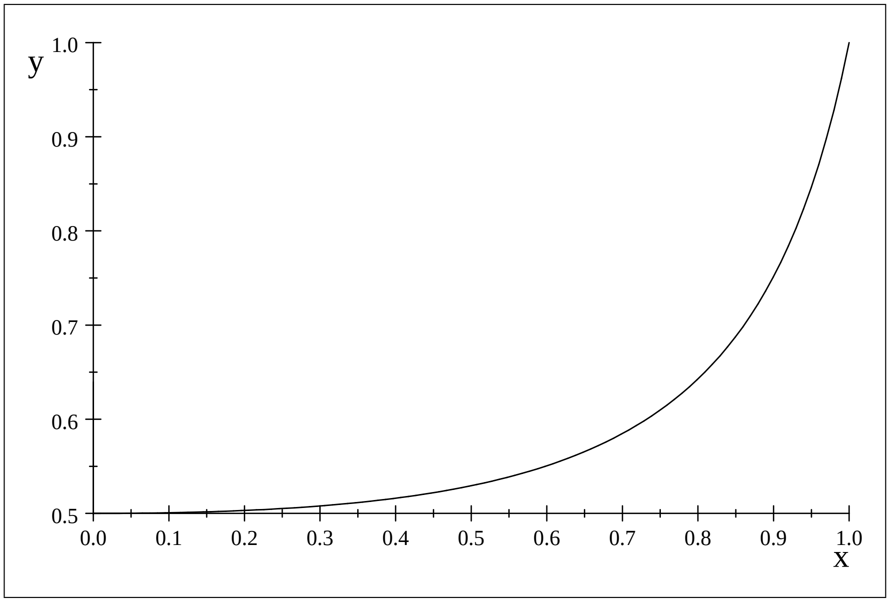
We can now plug in these profit level into the formula for the critical discount factor,

$$\begin{aligned}
\delta^{crit} &= \frac{\pi^r - \pi^m}{\pi^r - \pi^b} \\
&= \frac{\frac{a^2(2-r)^2}{16b(1-r)^2} - \frac{a^2}{4b(1-r)}}{\frac{a^2(2-r)^2}{16b(1-r)^2} - \frac{a^2}{b(2-r)^2}} \text{ (substituting in profit expressions)} \\
&= \frac{\frac{(2-r)^2}{16(1-r)^2} - \frac{1}{4(1-r)}}{\frac{(2-r)^2}{16(1-r)^2} - \frac{1}{(2-r)^2}} \text{ (cancel out } a^2/b\text{)} \\
&= \frac{\frac{(2-r)^2}{16(1-r)^2} - \frac{4(1-r)}{16(1-r)^2}}{\frac{(2-r)^4}{16(1-r)^2(2-r)^2} - \frac{16(1-r)^2}{16(1-r)^2(2-r)^2}} \text{ (to common denom.)} \\
&= \frac{(2-r)^2 - 4(1-r)}{\frac{(2-r)^4 - 16(1-r)^2}{(2-r)^2}} \text{ (cancelling terms)} \\
&= \frac{r^2}{\frac{(2-r)^4 - 16(1-r)^2}{(2-r)^2}} \text{ (simplifying numerator)} \\
&= \frac{r^2(2-r)^2}{(2-r)^4 - 16(1-r)^2} \text{ (rearraning)}
\end{aligned}$$

Thus we find that

$$\delta^{crit} = \frac{r^2(2-r)^2}{(2-r)^4 - 16(1-r)^2}.$$

How does the critical discount factor depend on r? The following figure plots δ^{crit} as a function of r over the relevant interval



It show that δ^{crit} increases in r; moreover, δ^{crit} limits to 0.5 when r goes to 0 and limits to δ^{crit} when r goes to 1.

In order to understand this we need to interpret r. Note that when r goes to 0, the coefficient c in the demand q_i on the rival firm's price p_j goes to zero. This means that the products in the limit are independent – their demands are not linked and each firm is a monopolist over its own product. In contrast, when r approaches unity, c approaches b. This means that the demand for firm i's output becomes highly sensitive to the price set by firm j for it's product; this represents the case where the two products become close substitutes.

The general conclusion is hence that it is difficult to sustain cooperation when the products are close substitutes.